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SPERNER'S THEOREM ON MAXIMAL-SIZED ANTICHAINS AND ITS GENERALIZATION

by

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1. Introduction

Let S be a set of n elements. Let $F_0 = \{A_1, A_2, \ldots, A_i, \ldots\}$ be a family of subsets of S such that no two subsets A_i and A_j in F_0 possess the property $A_i \supset A_j$. Sperner [1] proved that

$$|F_0| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}^{\dagger}$$

Sperner's result was generalized by Erdös [2] who showed that the size of a family of subsets of S such that no $\ell+1$ subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_{\ell+1}}$ in the family form a chain $A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_{\ell+1}}$ is upperbounded by the sum of the ℓ largest binomial coefficients of order n. Kleitman [3] and Katona [4] improved Sperner's result in the following way: Let $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$ be a partition of S. Let $F_0 = \{A_1, A_2, \ldots, A_i, \ldots\}$ be a family of subsets of S such that no two subsets A_i and A_j in F_0 possess the properties:

(i) $(A_i \cap S_1) = (A_j \cap S_1)$ and $(A_i \cap S_2) \supset (A_j \cap S_2)$; or

(ii) $(A_i \cap S_1) \supset (A_j \cap S_1)$ and $(A_i \cap S_2) = (A_j \cap S_2)$

Then

$$|F_0| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

DeBruijn, Tengbergen, and Kruyswijk [4] generalized Sperner's result in another direction: Let N be a positive integer and let $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ be a prime factor decomposition of N. We define the <u>degree</u> of an integer to be the sum of the exponents of the prime

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$^\dagger \lfloor k \rfloor$ denotes the largest integer not larger than k.

factors of the integer. We use the notation $S_m^i(N)$ to denote the number of divisors of N that are of degree i, where m is the degree of N. We follow the convention that $S_m^i(N) = 1$ for $i = 0$ and $S_m^i(N) = 0$ for $i < 0$ and $i > m$. Let $F_0 = \{a_1, a_2, \ldots, a_i, \ldots\}$ be a set of divisors of N such that no two divisors a_i and a_j in F possess the property $a_i | a_j$. According to DeBruijn, Tengbergen, and Kruyswijk,

$$|F_0| \leq S_m^{\lfloor m/2 \rfloor}(N)$$

Recently, the result of DeBruijn, Tengbergen, and Kruyswijk was generalized by Schönheim [6].

Katona [7] obtained a general result that includes all the results mentioned above as special cases. In this paper, we present a result that is similar to that of Katona's. Our result is more general in that it is applicable to the direct products of arbitrary partially ordered sets, yet Katona's result is restricted to direct products of "symmetrical chain graphs". For the lattice of subsets of a set and the lattice of divisors of an integer, we obtain simple proofs of many of the known results, and also are able to sharpen some of them. Our main result is Theorem 3. However, for exposition purpose we show first some special cases.

2. Product of partially ordered sets

Let $P = \{p_1, p_2, \ldots, p_i, \ldots\}$ be an arbitrary partially ordered set. Let L_h denote the set of integers $\{0,1,2,\ldots,h\}$ ordered by the "larger than or equal to" relation. Let $R = P \times L_h$. The elements in R shall be denoted (p_i, j), where $p_i \in P$ and $j \in \{0,1,2,\ldots,h\}$. Let

$T \subseteq R$. An element (p_i,j) in T is said to be an unobstructed element in T if there is no other element (p_i',j') in T such that $p_i > p_i'$ and $j = j'$ or $p_i = p_i'$ and $j > j'$. We also introduce the notation, for a fixed j,

$$T(j) = \{(p_i,j) \mid (p_i,j) \in T\}$$

$$T(\bar{j}) = \{(p_i,j') \mid (p_i,j') \in T, \quad j' > j\}$$

$$\bar{T}(\bar{j}) = \{(p_i,j) \mid (p_i,j') \in T, \quad j' > j\}$$

We shall use $P[\ell]$ to denote a maximal-sized subset of P such that no $\ell+1$ elements in the subset form a chain of length $\ell+1$. It will be understood that $P[\ell] = \emptyset$ for $\ell \leq 0$.

Theorem 1: Let $R = P \times L_h$. Let F_0 be a subset of R such that no $\ell+1$ elements in F_0 form a chain of length $\ell+1$. Then for $\ell > h$

$$|F_0| \leq \sum_{t=0}^{h} |P[\ell+h-2t]|$$

for $\ell \leq h$

$$|F_0| \leq \sum_{t=0}^{\ell-1} |P[\ell+h-2t]|$$

Proof: We prove the case $\ell > h$. The case $\ell \leq h$ can be proved in a similar manner. Let M_0 denote the set of all unobstructed elements in F_0. We note that $(p_i,j) \in M_0(\bar{0})$ implies that $(p_i,j') \notin M_0(\bar{0})$ for $j' \neq j$. Therefore, there is a one-to-one correspondence between the elements in $M_0(\bar{0})$ and $\bar{M}_0(\bar{0})$. That is,

$$|M_0(\bar{0})| = |\bar{M}_0(\bar{0})|$$

We also note that $(p_i, j) \in M_0(\bar{0})$ implies that $(p_i, 0) \notin F_0$. That is, the sets $\bar{M}_0(\bar{0})$ and $F_0(0)$ are disjoint. Consequently, we have

$$|F_0| = |F_0(0) \cup \bar{M}_0(\bar{0})| + |F_0 - F_0(0) - M_0(\bar{0})|$$

Recursively, for $t = 1,2,\ldots,h$ let F_t denote the set $F_{t-1} - F_{t-1}(t-1) - M_{t-1}(\overline{t-1})$ and M_t denote the set of unobstructed elements in F_t. For $t = 1,2,\ldots,h-1$, because

(i) $|M_t(\bar{t})| = |\bar{M}_t(\bar{t})|$

(ii) the sets $\bar{M}_t(\bar{t})$ and $F_t(t)$ are disjoint

we have

$$|F_t| = |F_t(t) \cup \bar{M}_t(\bar{t})| + |F_t - F_t(t) - M_t(\bar{t})|$$

It follows that

$$|F_0| = \sum_{t=0}^{h-1} |F_t(t) \cup \bar{M}_t(\bar{t})| + |F_h|$$

Since the set $F_0(0)$ does not contain chains of length larger than ℓ and the set $\bar{M}_0(\bar{0})$ does not contain chains of length larger than h, we have

$$|F_0(0) \cup \bar{M}_0(\bar{0})| \leq |P[\ell+h]|$$

Similarly, for $t = 1,2,\ldots,h-1$, since the set $F_t(t)$ does not contain chains of length larger than $\ell-t$[†] and the set $\bar{M}_t(\bar{t})$ does not contain

[†] Because every element of $F_t(t)$ dominates a chain of length t.

chains of length larger than $h-t$[†], we have

$$|F_t(t) \cup \overline{M}_t(\overline{t})| \leq |P[\ell+h-2t]|$$

Finally, because F_h does not contain chains of length larger than ℓ-h, we have

$$|F_h| \leq |P[\ell-h]|$$

We thus obtain

$$|F_0| \leq \sum_{t=0}^{h} |P[\ell+h-2t]| \quad \blacksquare$$

<u>Corollary 1.1</u>: Let $R = P \times L_1$. Let F_0 be a subset of R such that no $\ell+1$ elements in F_0 form a chain of length $\ell+1$. Then

$$|F_0| \leq |P[\ell+1]| + |P[\ell-1]|$$

From Corollary 1.1 we obtain Erdös' extension of Sperner's result:

<u>Corollary 1.2</u>: In the lattice of subsets of a finite set of n elements, the size of a family of subsets that does not contain a chain of length $\ell+1$ is upperbounded by the sum of the ℓ largest binomial coefficients of order n.

<u>Proof</u>: Let $R = (L_1)^{n-1} \times L_1$. We can then prove the corollary by induction on n, using the fact that the sum of the $\ell+1$ largest binomial coefficients of order n-1 and the ℓ-1 largest binomial

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† If (p_i,j) and (p_i',j) are two elements in $M_t(\overline{t})$, then they must be incomparable. Consequently, the corresponding elements (p_i,t) and (p_i',t) in $\overline{M}_t(\overline{t})$ must also be incomparable.

coefficients of order n-1 is equal to the sum of the ℓ largest binomial coefficients of order n.[†] ∎

Similarly, we can obtain Schönheim's extension of DeBruijn, Tengbergen, and Kruyswijk's result.

Corollary 1.3: In the lattice of divisors of an integer N, the size of a family of divisors that does not contain a chain of length $\ell+1$ is upperbounded by the sum of the ℓ largest values of $S_m^i(N)$, where m is the degree of N.

Proof: Let $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{n-1}^{\alpha_{n-1}} p_n^{\alpha_n}$. Let $R = (L_{\alpha_1} \times L_{\alpha_2} \times \dots \times L_{\alpha_{n-1}}) \times L_{\alpha_n}$. We can prove the corollary by induction on n, using the result that the sum of the ℓ largest values of $S_{m+\alpha}^i(Np^\alpha)$ is equal to

$$\sum_{t=0}^{\alpha} [\text{sum of the } \ell+\alpha-2t \text{ values of } S_m^i(N)]$$

for $\ell > \alpha$, and is equal to

$$\sum_{t=0}^{\ell-1} [\text{sum of the } \ell+\alpha-2t \text{ values of } S_m^i(N)]$$

for $\ell \leq \alpha$. This result can be obtained routinely from the facts:

(i) $S_m^1(N), S_m^2(N), \dots, S_m^{\lfloor m/2 \rfloor}(N)$ is a non-decreasing sequence

(ii) $S_m^j(N) = S_m^{m-j}(N)$

(iii) $S_{m+\alpha}^j(Np^\alpha) = S_m^j(N) + S_m^{j-1}(N) + S_m^{j-2}(N) + \dots + S_m^{j-\alpha}(N)$ ∎

Examining the proof of Theorem 1, we realize that the condition stated in Theorem 1 can be weakened:

[†] This result comes directly from the relation: $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$

Theorem 2: Let $R = P \times L_h$. Let F_0 be a subset of R such that there are no $\ell+1$ elements $(p_{i_1}, j_1), (p_{i_2}, j_2), \ldots, (p_{i_\ell}, j_\ell)$, $(p_{i_{\ell+1}}, j_{\ell+1})$ in F_0 possessing the properties: for some t, $0 \leq t \leq h$,

(i) $p_{i_1} > p_{i_2} > \ldots > p_{i_{\ell-t+1}}$ and

$j_1 = j_2 = \ldots = j_{\ell-t+1} = t$;

(ii) for $k = \ell-t+1, \ldots, \ell$, either $p_{i_k} > p_{i_{k+1}}$ and

$j_k = j_{k+1}$, or $p_{i_k} = p_{i_{k+1}}$ and $j_k > j_{k+1}$.

Then for $\ell > h$

$$|F_0| \leq \sum_{t=0}^{h} |P[\ell+h-2t]|$$

for $\ell \leq h$

$$|F_0| \leq \sum_{t=0}^{\ell-1} |P[\ell+h-2t]|$$

Corollary 2.1: Let $F_0 = \{A_1, A_2, \ldots, A_i, \ldots\}$ be a family of subsets in the lattice of subsets of a finite set S of n elements. Suppose that there are no $\ell+1$ subsets $A_{i_1}, A_{i_2}, \ldots, A_{i_{\ell+1}}$ in F_0 such that either $A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_{\ell+1}}$ and they all contain a certain element a, $a \in S$; or $A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_{\ell+1}}$ and they all do not contain the element a; or $A_{i_1} \supset A_{i_2} \supset \ldots \supset A_{i_\ell}$ and they all contain the element a and $A_{i_{\ell+1}} = A_{i_\ell} - \{a\}$. Then the size of F_0 is upperbounded by the sum of the ℓ largest binomial coefficients of order n.

Corollary 2.1 sharpens Erdös' result. If we set $\ell = 1$ in Corollary 2.1 we obtain a special case of Kleitman and Katona's result. Specifically, this case corresponds to partitioning S into S_1 and S_2 such that $S_1 = \{a\}$ and $S_2 = S - \{a\}$. We shall present a more general result in Corollary 3.1.

Similarly, Schönheim's result as stated in Corollary 1.3 can be sharpened:

<u>Corollary 2.2</u>: Let $F_0 = \{a_1, a_2, \ldots, a_i, \ldots\}$ be a collection of integers in the lattices of divisors of an integer $N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; \ldots \; p_n^{\alpha_n}$ such that there are no $\ell+1$ divisors $a_{i_1}, a_{i_2}, \ldots, a_{i_{\ell+1}}$ in F_0 possessing the properties: for some t, $0 \leq t \leq \alpha_n$,

(i) $a_1 \div a_2 \div \ldots \div a_{\ell-t+1}$ † and they all contain exactly the t^{th} power of p_n for $0 \leq t \leq \alpha_n$;

(ii) for $k = \ell-t+1, \ldots, \ell$, $a_{i_k} \div a_{i_{k+1}}$ and the quotient of a_{i_k} divided by $a_{i_{k+1}}$ either does not contain a power of p_n or is only a power of p_n.

Then the size of F_0 is upperbounded by the sum of the ℓ largest values of $S_m^i(N)$ where m is the degree of N.

Let P and Q be two partially ordered sets. Let $R = P \times Q$. Suppose that the elements in Q are partitioned into d disjoint chains.

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† We write $a_i \div a_j$ to mean a_i is divisible by a_j.

Let $h_1, h_2, \ldots, h_d$ denote the lengths of these chains. A direct consequence of Theorem 2 is

<u>Theorem 3</u>: Let F_0 be a subset of R such that no $\ell+1$ elements in F_0, (p_{i_1}, q_{j_1}), (p_{i_2}, q_{j_2}), ..., $(p_{i_{\ell+1}}, q_{j_{\ell+1}})$ possess the properties:

(i) $(p_{i_1},q_{j_1}) > (p_{i_2},q_{j_2}) > \ldots > (p_{i_{\ell+1}},q_{j_{\ell+1}})$;

(ii) $q_{j_1}, q_{j_2}, \ldots, q_{j_{\ell+1}}$ are in the same chain in the partition of Q into disjoint chains;

(iii) if in the chain containing q_{j_1}, q_{j_1} is the t^{th} element,† $q_{j_1} = q_{j_2} = \ldots = q_{j_{\ell-t+1}}$;

(iv) for $k = \ell-t+1, \ldots, \ell$, either $p_{i_k} > p_{i_{k+1}}$ and $q_{j_k} = q_{j_{k+1}}$, or $p_{i_k} = p_{i_{k+1}}$ and $q_{j_k} > q_{j_{k+1}}$.

Then

$$|F_0| \leq \sum_{i=1}^{d} Z_i$$

where for $\ell > h_i$

$$Z_i = \sum_{t=0}^{h_i} |P[\ell+h_i-2t]|$$

for $\ell \leq h_i$

$$Z_i = \sum_{t=0}^{\ell-1} |P[\ell+h_i-2t]| \quad .$$

In order to apply Theorems 3 to the lattice of subsets of a finite set, we define a <u>canonical partition</u> of $(L_1)^n$ into disjoint

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†The elements in a chain are labelled as the 0^{th}, 1st, 2nd, ..., t^{th}, ... elements, starting at the bottom of the chain.

chains† recursively as follows:

(i) L_1 is partitioned into a chain of length 2.

(ii) Let $p_{i_1} > p_{i_2} > \dots > p_{i_m}$ be a chain in a canonical partition of $(L_1)^{n-1}$. Then

$$(p_{i_1},1) > (p_{i_1},0) > (p_{i_2},0) > \dots > (p_{i_{m-1}},0) > (p_{i_m},0)$$

$$(p_{i_2},1) > (p_{i_3},1) > \dots > (p_{i_m},1)$$

will be two chains in a canonical partition of $(L_1)^n$.

Let $P = (L_1)^k$, $Q = (L_1)^n$, and $R = P \times Q$. By induction on n, we can immediately show that corresponding to a canonical partition of Q into disjoint chains, the sum $\sum_{i=1}^{d} Z_i$ in Theorem 3 is equal to the sum of the largest ℓ binomial coefficients of order k+n. We thus obtain:

<u>Corollary 3.1</u>: Let S be a finite set of size n. Suppose S is partitioned into S_1 and S_2 such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. Let us partition the subsets of S_2 into disjoint chains according to a canonical partition. Let F be a family of subsets of S such that there are no $\ell+1$ subsets $A_{i_1}, A_{i_2}, \dots, A_{i_{\ell+1}}$ in F possessing the properties:

(i) $A_{i_1} \supset A_{i_2} \supset \dots \supset A_{i_{\ell+1}}$

(ii) $A_{i_1} \cap S_1, A_{i_2} \cap S_2, \dots, A_{i_{\ell+1}} \cap S_2$ are in the same chain in the canonical partition of Q.

(iii) if in the chain containing $A_{i_1} \cap S_2$, $A_{i_1} \cap S_2$ is the t^{th} element, $A_{i_1} \cap S_2 = A_{i_2} \cap S_2 = \dots = A_{i_{\ell-t+1}} \cap S_2$

† These indeed are what are known as <u>symmetric chains</u> defined by DeBruijn, Tengbergen, and Kruyswijk.

(iv) for $k = \ell-t+1, \ldots, \ell$, either

$$A_{i_k} \cap S_1 \supset A_{i_{k+1}} \cap S_1 \text{ and } A_{i_k} \cap S_2 = A_{i_{k+1}} \cap S_2, \quad \text{or}$$

$$A_{i_k} \cap S_1 = A_{i_{k+1}} \cap S_1 \text{ and } A_{i_k} \cap S_2 \supset A_{i_{k+1}} \cap S_2.$$

Then the size of F_0 is upperbounded by the sum of the ℓ largest binomial coefficients of order n.

For $\ell = 1$, Corollary 3.1 is reduced to: Let F_0 be a family of subsets of S such that no two subsets A_i and A_j in F_0 possess the properties

(i) $A_i \cap S_2$ and $A_j \cap S_2$ are in the same chain in a canonical partition of the lattice of subsets of S_2; and

(ii) either $(A_i \cap S_1) \supset (A_j \cap S_1)$ and $(A_i \cap S_2) = (A_j \cap S_2)$, or $(A_i \cap S_1) = (A_j \cap S_1)$ and $(A_i \cap S_2) \supset (A_j \cap S_2)$.

Then the size of F_0 is upperbounded by the largest binomial coefficient of order n. Clearly, Theorem 3 also leads to a generalization of DeBruijn, Tengbergen, and Kruyswijk's result in the same sense as Corollary 3.1 generalizes Sperner's result. We shall leave the details to the reader.

3. Remarks

It is interesting to note that our approach is quite similar to Kleitman's approach [8] in solving Littlewood and Offord's problem on the distributions of linear combinations of vectors.

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15. Supplementary Notes

16. Abstracts

This paper generalizes previous work by Sperner [1], Erdös [2], Kleitman [3], Katona [4], DeBruijn, Tengbergen, and Kruyswijk [5], and Schönheim [6]. Such generalization not only leads to simple proofs of known results but also sharpens some of these results.

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